Lecture 31

Measure and Integration

I. K. Rana

17/3/11

$E \in \mathcal{L}_{\mathbb{R}^2}$

$\underline{x} = (x, y)$, $\underline{x} \cdot E = \{(ax, by) \mid (a,b) \in E\}$

$$\boxed{\forall E \in \mathcal{L}_{\mathbb{R}^2} \Longrightarrow \underline{x} \cdot E \in \mathcal{L}_{\mathbb{R}^2}}$$

$\mathcal{A} := \{E \in \mathcal{L}_{\mathbb{R}^2} \mid \underline{x} \cdot E \in \mathcal{L}_{\mathbb{R}^2}\}$

(i) $\mathcal{A}$ is a σ-algebra

$(\underline{x} \cdot E)^c = \underline{x} \cdot E^c \in \mathcal{L}_{\mathbb{R}^2}$

$\Rightarrow E^c \in \mathcal{L}_{\mathbb{R}^2}$

(ii) $E \times F$, $E, F \in \mathcal{L}_{\mathbb{R}}$

$$\underline{x}\cdot(E\times F) = (xE)\times(yF) \in \mathcal{L}_{\mathbb{R}^2}$$

$$\Rightarrow \quad \mathcal{L}_{\mathbb{R}}\times\mathcal{L}_{\mathbb{R}} \subseteq \underline{\mathcal{L}_{\mathbb{R}^2}}$$

$$\Rightarrow \quad \underbrace{\mathcal{L}_{\mathbb{R}}\otimes\mathcal{L}_{\mathbb{R}}} \subseteq \mathcal{L}_{\mathbb{R}^2} \longrightarrow$$

<u>Also</u> $\Rightarrow \quad E \in \mathbb{R}^2, \quad \lambda^*_{\mathbb{R}^2}(E) = 0$

$$\Rightarrow \lambda^*_{\mathbb{R}^2}(\underline{x}\cdot E) = 0 \longrightarrow$$

$$\Rightarrow \quad \mathcal{A} = \mathcal{L}_{\mathbb{R}^2}$$

<u>Claim</u> $\lambda_{\mathbb{R}^2}(\underline{x} \cdot E) = |xy| \lambda_{\mathbb{R}^2}(E)$

$\mathcal{M} := \{ E \in \mathcal{L}_{\mathbb{R}^2} \mid \text{Claim holds for } E \}$

(i) $\mathcal{M}$ is a monotone class

(2) $\mathcal{L}_{\mathbb{R}} \times \mathcal{L}_{\mathbb{R}} \subseteq \mathcal{M}$

(3) $\mathcal{M}$ is closed under finite disjoint unions

$\overset{(Ex)}{\Longrightarrow}$ $\mathcal{M} = \mathcal{L}_{\mathbb{R}^2}$

$$\because \quad \mathcal{L}_{\mathbb{R}} \times \mathcal{L}_{\mathbb{R}} \subseteq \mathcal{M}$$

$$\overset{(3)}{\Longrightarrow} \quad \mathcal{F}(\mathcal{L}_{\mathbb{R}} \times \mathcal{L}_{\mathbb{R}}) \subseteq \mathcal{M}$$

$$\overset{(1)}{\Longrightarrow} \quad \mathcal{M}(\mathcal{F}(\mathcal{L}_{\mathbb{R}} \times \mathcal{L}_{\mathbb{R}})) \subseteq \mathcal{M}$$

$$\| $$

$$\mathcal{L}_{\mathbb{R}} \otimes \mathcal{L}_{\mathbb{R}} \subseteq \mathcal{M}.$$

$$\Longrightarrow \quad \mathcal{L}_{\mathbb{R}^2} \subseteq \mathcal{M}$$

$\boxed{\cdot}$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

<u>Linear transformation</u>

(i) $T(\alpha \underline{x}) = \alpha T(\underline{x}) \quad \forall\ \alpha \in \mathbb{R},\ \underline{x} \in \mathbb{R}^2$

(ii) $T(\underline{x} + \underline{y}) = T(\underline{x}) + T(\underline{y}) \quad \forall\ \underline{x}, \underline{y} \in \mathbb{R}^2$

$$T \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} : A$$

$$T\underline{x} = \underline{A \begin{bmatrix} a \\ b \end{bmatrix}} \qquad \underline{x} = (a, b)$$

$$T \longleftrightarrow A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$T(x,y) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} ax \\ by \end{bmatrix}$$

$$= \underline{x} \cdot (x,y)$$

$$E \subseteq \mathbb{R}^2, \quad \underline{x} = (x,y)$$

$$\underline{x} \cdot E = T(E).$$

$$\lambda_{\mathbb{R}^2}(\underline{x}\cdot E) = |xy|\,\lambda_{\mathbb{R}^2}(E)$$

$$\underline{x} = (x,y)$$

$$x\cdot E = T(E)$$

$$T = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\det(T) = ab$$

$$\lambda_{\mathbb{R}^2}(\overset{T(E)}{\overset{\shortparallel}{\underline{x}\cdot E}}) = |\det T|\,\lambda_{\mathbb{R}^2}(E)$$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\underline{x} = (x, y) \longrightarrow T(\underline{x}) = (ax, by)$$

If either $a = 0$ or $b = 0$

$\det(T) = 0 \quad (= ab)$

i.e. T is Singular (it is not one-one.

$\Longrightarrow$ $T(\mathbb{R}^2)$ is a subspace of $\mathbb{R}^2$ of $\dim(T(\mathbb{R}^2)) \leq 1$.

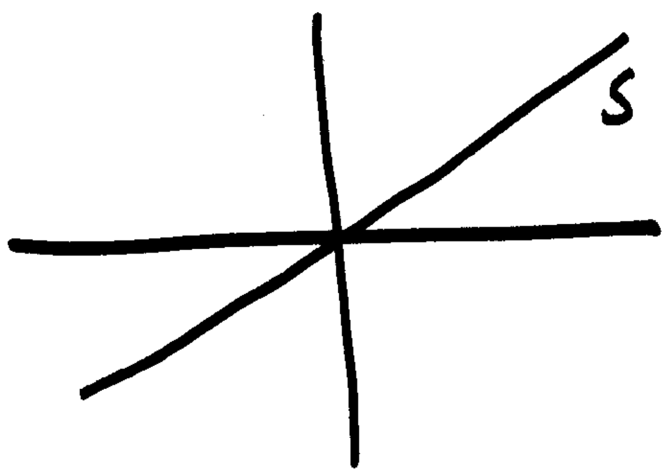
$S \subsetneq \mathbb{R}^2$, S a subspace,

$\dim(S) \leq 1$.

(i) $\dim(S) = 0$, $\Rightarrow$ $S = \{\underline{0}\}$

(ii) $\dim(S) = 1$, $\Rightarrow$ S is a line through the origin:

$$S = \{(x,y) \mid y = mx\} \text{ for some } m.$$



$\lambda_{\mathbb{R}^2}(S) \underset{(Ex)}{=} 0$

If T is singular

$\forall \ E \subseteq \mathbb{R}^2$

$T(E) \subseteq S, \ \dim(S) \leq 1$

$\Rightarrow \lambda_{\mathbb{R}^2}(T(E)) = 0$

$|\det(T)| = 0$

$\Rightarrow \quad T$ singular,

$\lambda_{\mathbb{R}^2}(T(E)) = 0 = |\det(T)| \lambda_{\mathbb{R}^2}(E)$

If T is nonsingular

$$T = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Rightarrow \text{neither } a, \text{ nor } b = 0.$$

$$|\det(T)| = |ab|$$

$$\Rightarrow \boxed{\begin{array}{l} \lambda_{\mathbb{R}^2}(T(E)) = |\det(T)|\,\lambda_{\mathbb{R}^2}(E) \\ \text{when } T \text{ is diagonal} \end{array}}$$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ non singular

__Claim__: $\forall\ E \in \mathcal{L}_{\mathbb{R}^2}$, $T(E) \in \mathcal{L}_{\mathbb{R}^2}$

and $\lambda_{\mathbb{R}^2}(T(E)) = |\det(T)|\,\lambda_{\mathbb{R}^2}(E)$?

__Case__: $E \in \mathcal{B}_{\mathbb{R}^2}$, $T(E) \in \mathcal{B}_{\mathbb{R}^2}$?

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, T linear

non-<u>singular</u> $\Longrightarrow$ T is bijective

and T is continuous.

(T^{-1} is also linear

and hence cont.)

$U \subseteq \mathbb{R}^2$, U open

$\Longrightarrow$ $T(U)$ is open

$\Longrightarrow$ $T(U) \in \mathcal{B}_{\mathbb{R}^2}$

$\mathcal{A} = \{ E \in \mathcal{B}_{\mathbb{R}^2} \mid T(E) \in \mathcal{B}_{\mathbb{R}^2} \}$

Opensets $\subseteq \mathcal{A}$.

Easy to check $\mathcal{A}$ is a σ-algebra

$\Longrightarrow$ $\mathcal{A} = \mathcal{B}_{\mathbb{R}^2}$

ie $\forall E \in \mathcal{B}_{\mathbb{R}^2} \Rightarrow T(E) \in \mathcal{B}_{\mathbb{R}^2}$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$\forall \ E \in \mathcal{B}_{\mathbb{R}^2}$, define

$$\mu_T(E) := \lambda_{\mathbb{R}^2}(T(E))$$

<u>Claim</u>: (i) μ_T is a measure: ✓

$$\mu_T\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \lambda_{\mathbb{R}^2}\left(T\left(\bigsqcup_{i=1}^{\infty} E_i\right)\right)$$

$$= \lambda_{\mathbb{R}^2}\left(\bigsqcup_{i=1}^{\infty} T(E_i)\right)$$

$$= \sum_{i=1}^{\infty} \lambda_{\mathbb{R}^2}(T(E_i))$$

$$= \sum_{i=1}^{\infty} \mu_T(E_i)$$

(ii) μ_T is translation-invariant:

$E \in \mathcal{B}_{\mathbb{R}^2}$, $\underline{x} \in \mathbb{R}^2$

$$\begin{aligned} \mu_T(E+\underline{x}) &= \lambda_{\mathbb{R}^2}(T(E+x)) \\ &= \lambda_{\mathbb{R}^2}(T(E)+T(x)) \\ &= \lambda_{\mathbb{R}^2}(T(E)) \\ &= \mu_T(E) \end{aligned}$$

(iii) $S = [0,1] \times [0,1] \in \mathcal{B}_{\mathbb{R}^2}$

$$\mu_T(S) = \lambda_{\mathbb{R}^2}(T[0,1] \times [0,1])$$

S is bounded
and hence $T(S)$ is also
bounded. with $\lambda_{\mathbb{R}^2}(T(S)) > 0$

$$\Longrightarrow \quad 0 < \mu_T(S) < +\infty.$$

$\Longrightarrow \quad \exists\ C(T) > 0$ such that

$$\mu_T(E) = C(T)\,\lambda_{\mathbb{R}^2}(E)$$

$$\forall\, E \in \mathcal{B}_{\mathbb{R}}$$

$\forall\ T$ nonsingular, $\exists\ C(T)$ such that

$$\lambda_{\mathbb{R}^2}(T(E)) = C(T)\,\lambda_{\mathbb{R}^2}(E)$$

Hence we have

$$T \longrightarrow C(T), \ \forall\, T$$

nonsingular

To show

$$C(T) = |\det(T)| \quad \forall\ T \text{ nonsingular}$$